

AN OPERATOR EXTENSION OF BOHR'S INEQUALITY

MOHAMMAD SAL MOSLEHIAN¹, JOSIP PEČARIĆ² AND IVAN PERIĆ³

ABSTRACT. We establish an operator extension of the following generalization of Bohr's inequality, due to M.P. Vasić and D.J. Kečkić:

$$\left| \sum_{i=1}^n z_i \right|^r \leq \left(\sum_{i=1}^n \alpha_i^{1/(1-r)} \right)^{r-1} \sum_{i=1}^n \alpha_i |z_i|^r \quad (r > 1, z_i \in \mathbb{C}, \alpha_i > 0, 1 \leq i \leq n).$$

We also present some norm inequalities related to our noncommutative generalization of Bohr's inequality.

1. INTRODUCTION

Let \mathfrak{A} be a C^* -algebra of Hilbert space operators and let T be a locally compact Hausdorff space. A field $(A_t)_{t \in T}$ of operators in \mathfrak{A} is called a continuous field of operators if the function $t \mapsto A_t$ is norm continuous on T . If μ is a Radon measure on T and the function $t \mapsto \|A_t\|$ is integrable, one can form the Bochner integral $\int_T A_t d\mu(t)$, which is the unique element in \mathfrak{A} such that

$$\varphi \left(\int_T A_t d\mu(t) \right) = \int_T \varphi(A_t) d\mu(t)$$

for every linear functional φ in the norm dual \mathfrak{A}^* of \mathfrak{A} ; cf. [3, Section 4.1].

Further, a field $(\varphi_t)_{t \in T}$ of positive linear mappings $\varphi : \mathfrak{A} \rightarrow \mathfrak{B}$ between C^* -algebras of operators is called continuous if the function $t \mapsto \varphi_t(A)$ is continuous for every $A \in \mathfrak{A}$. If the C^* -algebras include the identity operators, denoted by the same I , and the field $t \mapsto \varphi_t(I)$ is integrable with integral I , we say that $(\varphi_t)_{t \in T}$ is unital.

The classical Bohr's inequality states that for any $z, w \in \mathbb{C}$ and any positive real numbers r, s with $\frac{1}{r} + \frac{1}{s} = 1$,

$$|z + w|^2 \leq r|z|^2 + s|w|^2.$$

2000 *Mathematics Subject Classification.* Primary 47A63; secondary 47B10, 47A30, 47B15, 15A60.

Key words and phrases. Bohr's inequality, operator norm, operator inequality, positive operator, Hilbert space.

This inequality admits the operator extension

$$|A + B|^2 \leq r|A|^2 + s|B|^2$$

for operators A, B in the algebra $\mathbb{B}(\mathcal{H})$ of all bounded linear operators on a complex Hilbert space \mathcal{H} (to see this use the Cauchy–Schwarz inequality and the fact that the operator C is positive if and only if $\langle Cx, x \rangle \geq 0$).

Over the years, interesting generalizations of this inequality have been obtained in various settings; cf. [2, 6, 7, 8, 9, 11]. There is one of special interest given by M.P. Vasić and D.J. Kečkić [10]:

If z_1, \dots, z_n are complex numbers, $r > 1$ and $\alpha_i > 0$ ($i = 1, 2, \dots, n$), then

$$\left| \sum_{i=1}^n z_i \right|^r \leq \left(\sum_{i=1}^n \alpha_i^{1/(1-r)} \right)^{r-1} \sum_{i=1}^n \alpha_i |z_i|^r. \quad (1.1)$$

This is indeed an immediate consequence of the Hölder inequality. In this paper we establish the operator version of inequality (1.1) and apply the obtained operator inequalities to obtain some norm inequalities related to our operator extension of Bohr’s inequality.

2. MAIN RESULTS

Recall that a continuous real function f defined on a real interval J of any type is said to be operator convex if $f(\lambda A + (1 - \lambda)B) \leq \lambda f(A) + (1 - \lambda)f(B)$ holds for all $\lambda \in [0, 1]$ and all self-adjoint operators A, B acting on a Hilbert space with spectra in J . For instance $f(x) = x^r$, where $1 \leq r \leq 2$ is operator convex on $[0, \infty)$; see [1, p. 123]. In [5], the authors gave a general formulation of Jensen’s inequality for unital fields of positive linear mappings in which they dealt with operator convex functions.

We need the main result [5, Theorem 2.1]. We state it for the sake of convenience.

Theorem 2.1. *Let f be an operator convex function on an interval J and let \mathfrak{A} and \mathfrak{B} be unital C^* -algebras. If $(\varphi_t)_{t \in T}$ is a unital field of positive linear mappings $\varphi_t : \mathfrak{A} \rightarrow \mathfrak{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ , then the inequality*

$$f \left(\int_T \varphi_t(A_t) d\mu(t) \right) \leq \int_T \varphi_t(f(A_t)) d\mu(t).$$

holds for every bounded continuous field $(A_t)_{t \in T}$ of self-adjoint elements of \mathfrak{A} with spectra contained in J .

Utilizing the theorem above we prove our main result.

Theorem 2.2. *Let \mathfrak{A} and \mathfrak{B} be C^* -algebras of operators containing I , let T be a locally compact Hausdorff space equipped with a bounded Radon measure μ , let (α_t) a bounded continuous nonnegative function such that $(\alpha_t) \in L^{\frac{1}{1-r}}(T, \mu)$ and $1 < r \leq 2$. Let $(A_t)_{t \in T}$ be a bounded continuous field of positive elements in \mathfrak{A} and let $(\varphi_t)_{t \in T}$ be a field of positive linear mappings $\varphi_t : \mathfrak{A} \rightarrow \mathfrak{B}$ defined on T satisfying*

$$\int_T \alpha_t^{1/(1-r)} \varphi_t(I) d\mu(t) \leq \int_T \alpha_t^{1/(1-r)} d\mu(t) I.$$

Then

$$\left(\int_T \varphi_t(A_t) d\mu(t) \right)^r \leq \left(\int_T \alpha_t^{1/(1-r)} d\mu(t) \right)^{r-1} \int_T \alpha_t \varphi_t(A_t^r) d\mu(t). \quad (2.1)$$

Proof. Let ∞ be an object not belonging to T . Consider $T_\infty := T \cup \{\infty\}$ as a locally compact topological space by equipping $\{\infty\}$ with the discrete topology and extend μ on T_∞ by $\mu(\{\infty\}) = 1$. Set $f(x) = x^r$ ($x \in [0, \infty)$), $\tilde{\varphi}_t := \frac{P_t}{Q} \varphi_t$ ($t \in T_\infty$), where $P_t := \alpha_t^{1/(1-r)}$, $P_\infty := 1$, $Q := \int_T \alpha_t^{1/(1-r)} d\mu(t)$ and

$$\varphi_\infty(A) := \langle Ae, e \rangle \left(\int_T P_t d\mu(t) I - \int_T P_t \varphi_t(I) d\mu(t) \right),$$

in which A belongs to the C^* -algebra \mathfrak{A} acting on a Hilbert space \mathcal{H} and $e \in \mathcal{H}$ is a fixed unit vector. Then

$$\begin{aligned} \int_{T_\infty} \tilde{\varphi}_t(I) d\mu(t) &= \frac{1}{Q} \int_T P_t \varphi_t(I) d\mu(t) + \tilde{\varphi}_\infty(I) \\ &= \frac{1}{Q} \int_T P_t \varphi_t(I) d\mu(t) + \frac{1}{Q} \left(\int_T P_t d\mu(t) I - \int_T P_t \varphi_t(I) d\mu(t) \right) \\ &= I. \end{aligned}$$

It follows from Theorem 2.1 that

$$\left(\int_{T_\infty} \tilde{\varphi}_t(\tilde{A}_t) d\mu(t) \right)^r \leq \int_{T_\infty} \tilde{\varphi}_t(\tilde{A}_t^r) d\mu(t)$$

namely

$$\left(\int_{T_\infty} P_t \varphi_t(\tilde{A}_t) d\mu(t) \right)^r \leq Q^{r-1} \int_{T_\infty} P_t \varphi_t(\tilde{A}_t^r) d\mu(t) \quad (2.2)$$

for all bounded continuous fields $(\tilde{A}_t)_{t \in T_\infty}$.

Put $\tilde{A}_t = A_t/P_t$ ($t \in T$) and $\tilde{A}_\infty = 0$ in (2.2) to obtain

$$\left(\int_T \varphi_t(A_t) d\mu(t) \right)^r \leq Q^{r-1} \int_T P_t^{1-r} \varphi_t(A_t^r) d\mu(t). \quad (2.3)$$

By the definitions of P_t and Q , (2.3) becomes (2.1). \square

Remark 2.3. Using analogous arguing as in the proof of Theorem 2.2, one can prove the following Jensen's inequality. Suppose that (φ_t) is a continuous field of positive linear mappings $\varphi_t : \mathfrak{A} \rightarrow \mathfrak{B}$, \mathfrak{A} and \mathfrak{B} are unital C^* -algebras, (A_t) is a bounded continuous field of self-adjoint elements in \mathfrak{A} with spectra in an interval J such that $0 \in J$, (β_t) is a continuous nonnegative function such that $\int_T \beta_t d\mu(t) > 0$ and

$$\int_T \beta_t \varphi_t(I) d\mu(t) \leq \int_T \beta_t d\mu(t) I. \quad (2.4)$$

If f is an operator convex function on an interval J such that $f(0) \leq 0$, then

$$f\left(\frac{1}{\int_T \beta_t d\mu(t)} \int_T \beta_t \varphi_t(A_t) d\mu(t)\right) \leq \frac{1}{\int_T \beta_t d\mu(t)} \int_T \beta_t \varphi_t(f(A_t)) d\mu(t). \quad (2.5)$$

If equality holds in (2.4), then it is not necessary to assume that $0 \in J$ and $f(0) \leq 0$.

A discrete version of the theorem above is the following result obtained by taking $T = \{1, \dots, n\}$.

Corollary 2.4. *Let $1 < r \leq 2$, $\alpha_i > 0$ ($i = 1, \dots, n$), let A_1, \dots, A_n be positive operators acting on a Hilbert space \mathcal{H} and let φ_i ($i = 1, \dots, n$) be positive linear mappings on $\mathbb{B}(\mathcal{H})$ satisfying*

$$\sum_{i=1}^n \alpha_i^{1/(1-r)} \varphi_i(I) \leq \sum_{i=1}^n \alpha_i^{1/(1-r)} I. \quad (2.6)$$

Then

$$\left(\sum_{i=1}^n \varphi_i(A_i)\right)^r \leq \left(\sum_{i=1}^n \alpha_i^{1/(1-r)}\right)^{r-1} \sum_{i=1}^n \alpha_i \varphi_i(A_i^r).$$

By setting $\varphi_i(A) = X_i^* A X_i$ in Corollary 2.4 we find that

Corollary 2.5. *Let $1 < r \leq 2$, $\alpha_i > 0$ for $i = 1, \dots, n$, let A_1, \dots, A_n be bounded operators acting on a Hilbert space \mathcal{H} with $A_i \geq 0$ and let $X_1, \dots, X_n \in \mathbb{B}(\mathcal{H})$ satisfying*

$$\sum_{i=1}^n \alpha_i^{1/(1-r)} X_i^* X_i \leq \sum_{i=1}^n \alpha_i^{1/(1-r)} I. \quad (2.7)$$

Then

$$\left(\sum_{i=1}^n X_i^* A_i X_i\right)^r \leq \left(\sum_{i=1}^n \alpha_i^{1/(1-r)}\right)^{r-1} \sum_{i=1}^n \alpha_i X_i^* A_i^r X_i.$$

Condition (2.7) trivially holds if $X_i^* X_i \leq I$ for all $i = 1, \dots, n$. In fact we can give another proof of the result in this case.

Corollary 2.6. *Let $A_1, \dots, A_n, X_1, \dots, X_n \in \mathbb{B}(\mathcal{H})$ with $A_i \geq 0$, $X_i^* X_i \leq I$ for $i = 1, \dots, n$, and let $1 < r \leq 2$, $\alpha_i > 0$ for $i = 1, \dots, n$. Then*

$$\left(\sum_{i=1}^n X_i^* A_i X_i \right)^r \leq \left(\sum_{i=1}^n \alpha_i^{1/(1-r)} \right)^{r-1} \sum_{i=1}^n \alpha_i X_i^* A_i^r X_i.$$

Proof. First note that $0 \leq (X_i^* A_i X_i)^r \leq X_i^* A_i^r X_i$ for each i ; cf. [4, Theorem 2.1]. For $i = 1, \dots, n$, set $\beta_i = \alpha_i^{1/(1-r)}$, $B_i = X_i^* A_i X_i / \beta_i$. We have

$$\begin{aligned} \left(\sum_{i=1}^n X_i^* A_i X_i \right)^r &= \left(\sum_{i=1}^n \beta_i B_i \right)^r \\ &= \left(\sum_{i=1}^n \beta_i \sum_{j=1}^n \frac{\beta_j}{\sum_{k=1}^n \beta_k} B_j \right)^r \\ &= \left(\sum_{i=1}^n \beta_i \right)^r \left(\sum_{i=1}^n \frac{\beta_i}{\sum_{k=1}^n \beta_k} B_i \right)^r \\ &\leq \left(\sum_{i=1}^n \beta_i \right)^r \frac{\sum_{i=1}^n \beta_i B_i^r}{\sum_{i=1}^n \beta_i} \\ &\quad \text{(by the operator convexity of } f(t) = t^r \text{)} \\ &= \left(\sum_{i=1}^n \beta_i \right)^{r-1} \sum_{i=1}^n \beta_i^{1-r} (X_i^* A_i X_i)^r \\ &= \left(\sum_{i=1}^n \alpha_i^{1/(1-r)} \right)^{r-1} \sum_{i=1}^n \alpha_i (X_i^* A_i X_i)^r \\ &\leq \left(\sum_{i=1}^n \alpha_i^{1/(1-r)} \right)^{r-1} \sum_{i=1}^n \alpha_i X_i^* A_i^r X_i. \end{aligned}$$

□

Proposition 2.7. *Let $A_1, \dots, A_n \in \mathbb{B}(\mathcal{H})$ with $A_i^* A_j = 0$ for $1 \leq i \neq j \leq n$, and let $2 < r \leq 4$, $\alpha_i > 0$ for $i = 1, \dots, n$. Then*

$$\left| \sum_{i=1}^n A_i \right|^r \leq \left(\sum_{i=1}^n \alpha_i^{2/(2-r)} \right)^{(r-2)/2} \sum_{i=1}^n \alpha_i |A_i|^r$$

and

$$\left\| \sum_{i=1}^n A_i \right\|^r \leq \left(\sum_{i=1}^n \alpha_i^{2/(2-r)} \right)^{(r-2)/2} \sum_{i=1}^n \alpha_i \|A_i\|^r. \quad (2.8)$$

Proof.

$$\begin{aligned}
\left| \sum_{i=1}^n A_i \right|^r &= \left(\left| \sum_{i=1}^n A_i \right|^2 \right)^{r/2} \\
&= \left(\sum_{i,j=1}^n A_i^* A_j \right)^{r/2} \\
&= \left(\sum_{i=1}^n |A_i|^2 \right)^{r/2} \\
&\leq \left(\sum_{i=1}^n \alpha_i^{2/(2-r)} \right)^{(r-2)/2} \sum_{i=1}^n \alpha_i (|A_i|^2)^{r/2} \quad (\text{by Corollary 2.6 with } X_i = I) \\
&= \left(\sum_{i=1}^n \alpha_i^{2/(2-r)} \right)^{(r-2)/2} \sum_{i=1}^n \alpha_i |A_i|^r.
\end{aligned}$$

Inequality (2.8) is easily deduced from the fact that $\|Z\|^r = \||Z|^r\|$ for each $Z \in \mathbb{B}(\mathcal{H})$. \square

Remark 2.8. It is clear that $A_1, \dots, A_n \in \mathbb{B}(\mathcal{H})$ have orthogonal ranges if and only if $A_i^* A_j = 0$. An example of such operators is obtained by considering an orthogonal family $(e_i)_{1 \leq i \leq n}$ and a vector x in \mathcal{H} and defining the rank one operators $A_i : \mathcal{H} \rightarrow \mathcal{H}$ by $A_i = e_i \otimes x$, $1 \leq i \leq n$. Then $A_i^* A_j = \langle e_j, e_i \rangle x \otimes x$ for all $1 \leq i, j \leq n$.

REFERENCES

- [1] R. Bhatia, *Matrix Analysis*, Springer-Verlag, New York, 1997.
- [2] W.-S. Cheung and J.E. Pečarić, *Bohr's inequalities for Hilbert space operators*, J. Math. Anal. Appl. **323** (2006), no. 1, 403–412.
- [3] F. Hansen and G.K. Pedersen, *Jensen's operator inequality*, Bull. London Math. Soc. **35** (2003), 553–564.
- [4] F. Hansen and G.K. Pedersen, *Jensen's inequality for operators and Löwner's theorem*, Math. Ann. **258** (1982), 229–241.
- [5] F. Hansen, J.E. Pečarić and I. Perić, *Jensen's operator inequality and its converses*, Math. Scand. **100** (2007), no. 1, 61–73.
- [6] O. Hirzallah, *Non-commutative operator Bohr inequality*, J. Math. Anal. Appl. **282** (2003), 578–583.
- [7] M.S. Moslehian, R. Rajić, *Generalizations of Bohr's inequality in Hilbert C^* -modules*, Linear Multilinear Algebra (to appear).
- [8] V.I. Paulsen and D. Singh, *Bohr's inequality for uniform algebras* Proc. Amer. Math. Soc. **132** (2004), no. 12, 3577–3579.

- [9] J.E. Pečarić and Th.M. Rassias, *Variations and generalizations of Bohr's inequality*, J. Math. Anal. Appl. **174** (1993), no. 1, 138–146.
- [10] M.P. Vasić and D.J. Kečkić, *Some inequalities for complex numbers*, Math. Balkanica **1** (1971), 282–286.
- [11] F. Zhang, *On the Bohr inequality of operators*, J. Math. Anal. Appl. **333** (2007), 1264–1271.

¹ DEPARTMENT OF PURE MATHEMATICS, FERDOWSI UNIVERSITY OF MASHHAD, P.O. Box 1159, MASHHAD 91775, IRAN;
CENTER OF EXCELLENCE IN ANALYSIS ON ALGEBRAIC STRUCTURES (CEAAS), FERDOWSI UNIVERSITY OF MASHHAD, IRAN.

E-mail address: moslehian@ferdowsi.um.ac.ir and moslehian@ams.org

² FACULTY OF TEXTILE TECHNOLOGY, UNIVERSITY OF ZAGREB, PIEROTTIJEVA 6, 10000 ZAGREB, CROATIA

E-mail address: pecaric@mahazu.hazu.hr

³ FACULTY OF FOOD TECHNOLOGY AND BIOTECHNOLOGY, MATHEMATICS DEPARTMENT, UNIVERSITY OF ZAGREB, PIEROTTIJEVA 6, 10000 ZAGREB, CROATIA

E-mail address: iperic@pbf.hr